Semi-implicit Time-stepping Algorithm used in MH4D for Simulations for Innovative Confinement Concept Fusion Devices

Srinath Vadlamani,

George Marklin, Uri Shumlak and Thomas Jarboe,

Roberto Lionello*

Plasma Science and Innovation Center, University of Washington,

*Science Applications International Corporation

Main Ideas

- PSI-Center: further developing a tetrahedral mesh MHD simulation code to model Emerging Concept (EC) experiments (MH4D: Lionello & Schnack, 2004)
  - represent the complex 3-dimensional geometry of many EC experiments
  - using existing programs for gridding, parallel implementation, and visualization
- Semi-implicit time-stepping algorithm validation
  - Sound wave in a box
  - Alfvén wave in a box

Code: Resistive MHD Model

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \qquad \mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B}$$

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J}$$

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \rho \mathbf{v}$$

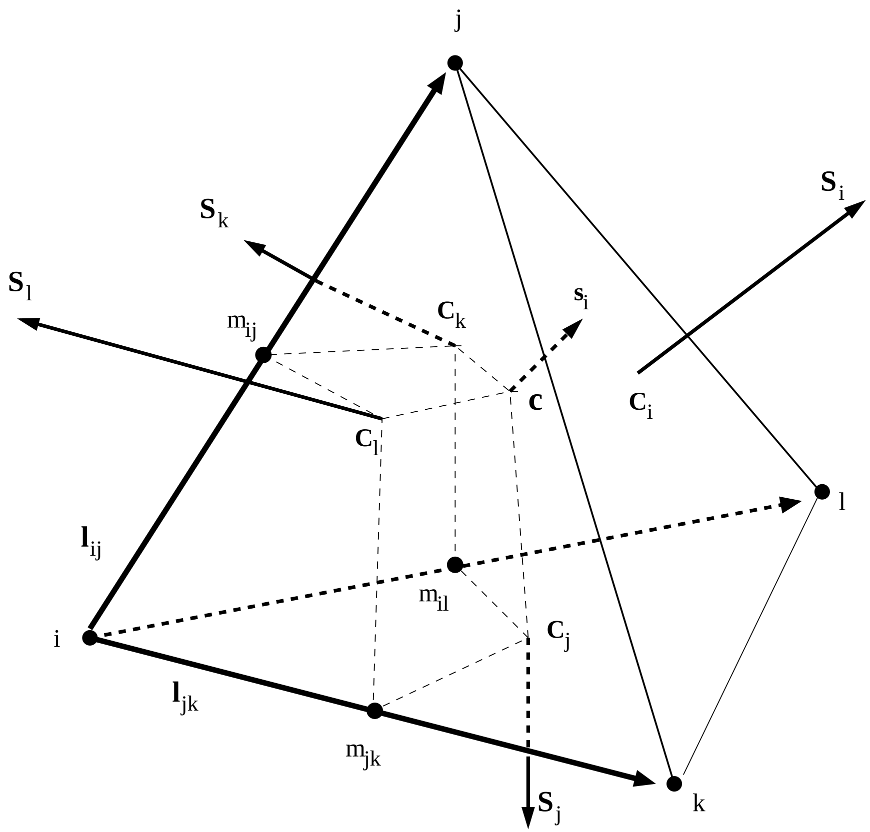
$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\boldsymbol{\nabla} p + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{v}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \mathbf{v} + \boldsymbol{\nabla} \cdot \mathbf{q} + Q$$

Alfvén velocity: $V_A = B/\sqrt{\rho}$
Alfvén transit time: $\tau_A = a/V_A$
Resistive diffusion time : $\tau_R = a^2/\eta$

Lundquist number: $S = \tau_R/\tau_A$

Tetrahedral Grid



- Sides are labeled by index of their opposite vertex
- **c**: centroid of tetrahedron
- m_{ij}: midpoint of edge $\mathbf{l}_{ij}$
- C_i: centroid of side i
- $\mathbf{S}_i$: vector area of side i
- $\mathbf{s}_i$: vector area of dual median surface
- V_c: volume of tetraderon c

$$\mathbf{s}_i = \frac{1}{3}\mathbf{S}_i \qquad V_c = \frac{1}{6}\mathbf{l}_{ij} \cdot (\mathbf{l}_{ik} \times \mathbf{l}_{il}) = -\frac{1}{3}\mathbf{l}_{ij} \cdot \mathbf{S}_j$$

Finite Volume Formalism

Integral relations to define differential operators

Gradient:

$$\int_V \nabla f dV = \int_S \hat{\mathbf{n}} f dS$$

Divergence:

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} dV = \int_S \hat{\mathbf{n}} \cdot \mathbf{F} dS$$

Curl in 2D

$$\int_s \mathbf{n} \cdot \boldsymbol{\nabla} \times \mathbf{F} dS = \oint_C \mathbf{F} \cdot d\mathbf{l}$$

Curl in 3D

$$\int_V \nabla \times \mathbf{F} dV = \int_S \hat{\mathbf{n}} \times \mathbf{F} dS$$

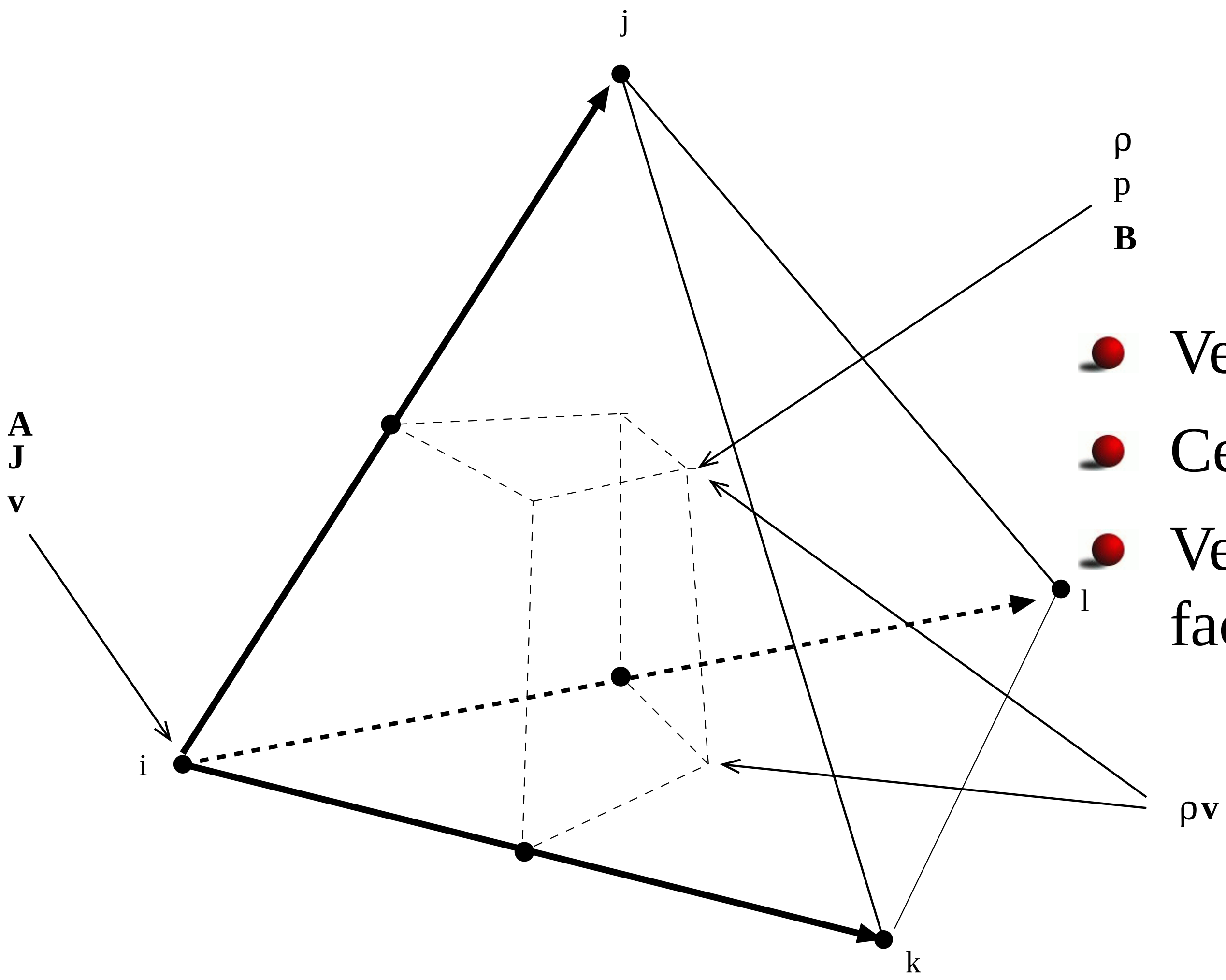
Boundary Condition

- Natural boundary condition,

$$\delta \mathbf{A}_t \cdot (\mathbf{B} \times \hat{\mathbf{n}}) = 0$$

- We must specify $\mathbf{A}_t = (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}) \cdot \mathbf{A}$ on the boundary (Dirichlet)
- Item for further PSI-center development

Variables on Grid



- Vertices: $\mathbf{A}$, $\mathbf{J}$, $\mathbf{v}$
- Centroids: B, ρ, p
- Velocity averaged to faces or centroids

Partial Implicit Time Stepping

Large spectrum of eigenvalues associated with MHD operators
$\Rightarrow$ large range of time scales

$$\frac{\partial u}{\partial t} = \underbrace{\mathcal{M}\{u\}}_{\text{Full MHD Operator}} = \underbrace{\overbrace{\mathcal{F}\{u\}}^{\text{Fast time scales}}}_{\text{Alfvén waves, soundwaves}} + \underbrace{\overbrace{\mathcal{S}\{u\}}^{\text{Slow time scales}}}_{\text{interesting physics}}$$

To avoid time step restriction, treat "fast" part of operator implicitly

$$\frac{u^{n+1} - u^n}{\Delta t} = F u^{n+1} + S u^n$$

Quite difficult to achieve precise discrete representation

No need for operator S

F and M are known operators in MHD computations

$$\begin{aligned}\frac{u^{n+1}-u^n}{\Delta t} &= Fu^{n+1} + \underbrace{(M-F)}_{S=M-F}u^n \\ &= Mu^n + \Delta t F\left(\frac{u^{n+1}-u^n}{\Delta t}\right)\end{aligned}$$

Works for arbitrary operator F

Semi-Implicit Method

For an arbitrary operator G

$$\left(I - \underbrace{\Delta t G}_{\text{SI operator}}\right) u^{n+1} = \underbrace{(I + \Delta t M)\, u^n}_{\text{Explicit}} - \underbrace{\Delta t G}_{\text{SI operator}} u^n$$

Want to choose G for ease of inversion and inclusion of modes of interest

Semi-implicit MHD operator

- use of linearized, ideal MHD wave equation

$$\rho_0 \frac{\partial^2 \mathbf{v}}{\partial t^2} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}_0) \times \mathbf{B}_0$$

- large spectrum of normal modes
- anisotropic spatial operator
- modify equation of motion

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} p + \mathbf{J} \times \mathbf{B} + \alpha \Delta t \mathbf{S}\left(\frac{\partial \mathbf{v}}{\partial t}\right)$$

where $\mathbf{S}(\mathbf{v}) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \times \mathbf{B}$

S is self-adjoint.

Discrete Semi-Implicit Operator

Use variational principle:

$$I(\mathbf{v}) = \frac{1}{2}\int \left[\left| \nabla \times (\mathbf{v} \times \mathbf{B})^2 \right| + 2\mathbf{S} \cdot \mathbf{v} \right] dV$$

Let:

$$\mathbf{v}\times\mathbf{B} = \sum_j (\mathbf{v}_j \times \mathbf{B}_j)\alpha_j(\mathbf{x}), \quad \mathbf{v} = \sum_j \mathbf{v}_j \alpha_j(\mathbf{x}), \quad \mathbf{S} = \sum_j \mathbf{S}_j \delta(\mathbf{x}-\mathbf{x}_j)$$

Discrete variational principle:

$$I(\mathbf{v}) = \frac{1}{2}\sum_k \sum_j \mathbf{v}_j \cdot \mathfrak{N}(j,k) \cdot \mathbf{v}_k + \mathbf{S}_k \cdot \mathbf{v}_k$$

where $\mathfrak{N}(j,k) = \mathbf{B}_j \times \mathfrak{M}(j,k) \times \mathbf{B}_k$ and is symmetric

Minimization of $I \rightarrow$ discrete semi-implicit operator $\mathbf{S}_i$:

$$\frac{\partial I}{\partial \mathbf{v}_i} = 0 \iff \mathbf{S}_i = -\sum_j \mathfrak{N}(i,j) \cdot \mathbf{v}_j, \quad \forall\, i.$$

Time Stepping Algorithm (I)

$$\frac{\mathbf{A}^* - \mathbf{A}^{n-1/2}}{\Delta t} = \frac{1}{2}\mathbf{v}^n \times \mathbf{B}^{n-1/2} - \frac{1}{2}\mathbf{v}^n \cdot \boldsymbol{\nabla}\mathbf{A}^{n-1/2}, \quad P \qquad (1)$$

$$\frac{\mathbf{A}^{n+1/2} - \mathbf{A}^{n-1/2}}{\Delta t} = \mathbf{v}^n \times \mathbf{B}^* - \frac{\eta}{2}\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A}^{n+1/2} - \frac{\eta}{2}\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A}^{n-1/2}, \quad C \qquad (2)$$

$$\frac{\rho^* - \rho^{n-1/2}}{\Delta t} = -\boldsymbol{\nabla} \cdot \left(\rho^{n-1/2}\mathbf{v}^n\right), \quad P \qquad (3)$$

$$\frac{\rho^{n+1/2} - \rho^{n-1/2}}{\Delta t} = -\boldsymbol{\nabla} \cdot \left(\rho^*\mathbf{v}^n\right), \quad C \qquad (4)$$

Time Stepping Algorithm (II)

$$\frac{p^* - p^{n-1/2}}{\Delta t} = -\boldsymbol{\nabla} \cdot \left(p^{n-1/2}\mathbf{v}^n\right), \qquad P \quad (5)$$

$$\frac{p^{n+1/2} - p^{n-1/2}}{\Delta t} = -\boldsymbol{\nabla} \cdot \left(p^*\mathbf{v}^n\right) - (\gamma - 1)\, p^{n-1/2}\boldsymbol{\nabla} \cdot \mathbf{v}^n, \qquad C \quad (6)$$

Semi-Implicit Momentum.

$$\frac{\mathbf{v}^* - \mathbf{v}^n}{\Delta t} = -\mathbf{v}^n \cdot \nabla \mathbf{v}^n, \qquad P \tag{7}$$

$$\frac{\mathbf{v}^{**} - \mathbf{v}^n}{\Delta t} = -\mathbf{v}^n \cdot \nabla \mathbf{v}^* \qquad C \tag{8}$$

$$+\frac{\mathbf{J}^{n+1/2} \times \mathbf{B}^{n+1/2}}{\rho^{n+1/2}} - \frac{\nabla p^{n+1/2}}{\rho^{n+1/2}}$$

$$+\frac{\nabla \cdot C^2 \Delta t^2 \rho_0^{n+1/2} \nabla \left(\mathbf{v}^{**} - \mathbf{v}^n\right)}{\Delta t \rho_0^{n+1/2}},$$

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^{**}}{\Delta t} = \frac{\nabla \nu \rho_o^{n+1/2} \nabla \mathbf{v}^{n+1}}{\rho_0^{n+1/2}}. \tag{9}$$

Linear Sound Wave

Linear system:

$$\frac{\partial \mathbf{v}_1}{\partial t} = \frac{1}{\rho_0}\boldsymbol{\nabla} P_1$$

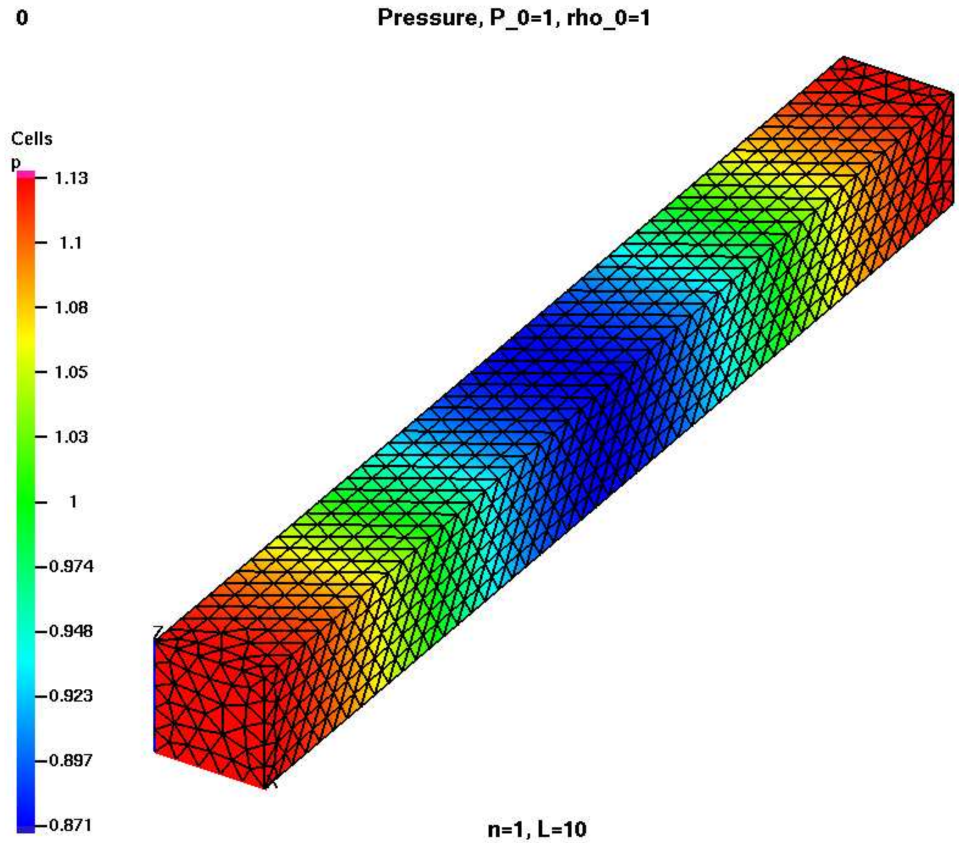
$$\frac{\partial P_1}{\partial t} = -\gamma P_0 \left(\boldsymbol{\nabla}\cdot\mathbf{v}_1\right)$$

Eigenvalue: $\omega = k\sqrt{\gamma\frac{P_0}{\rho_0}}$

Eigenvectors:

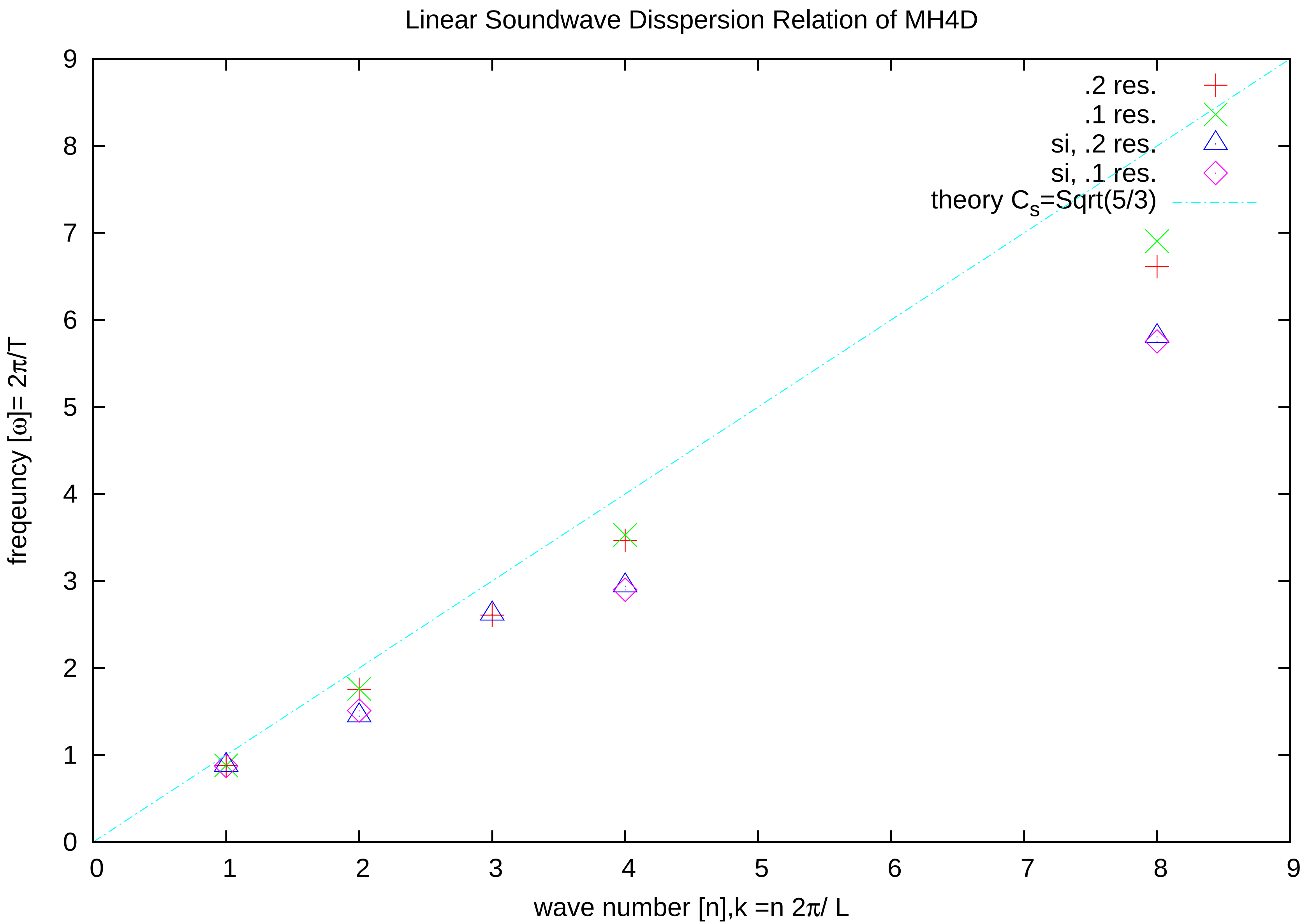
$$\mathbf{v}_1 = \epsilon\sin(ky)\sin(\omega t)\hat{\mathbf{y}}$$

$$P_1 = \epsilon\sqrt{\gamma P_0\rho_0}\cos(ky)\cos(\omega t)$$

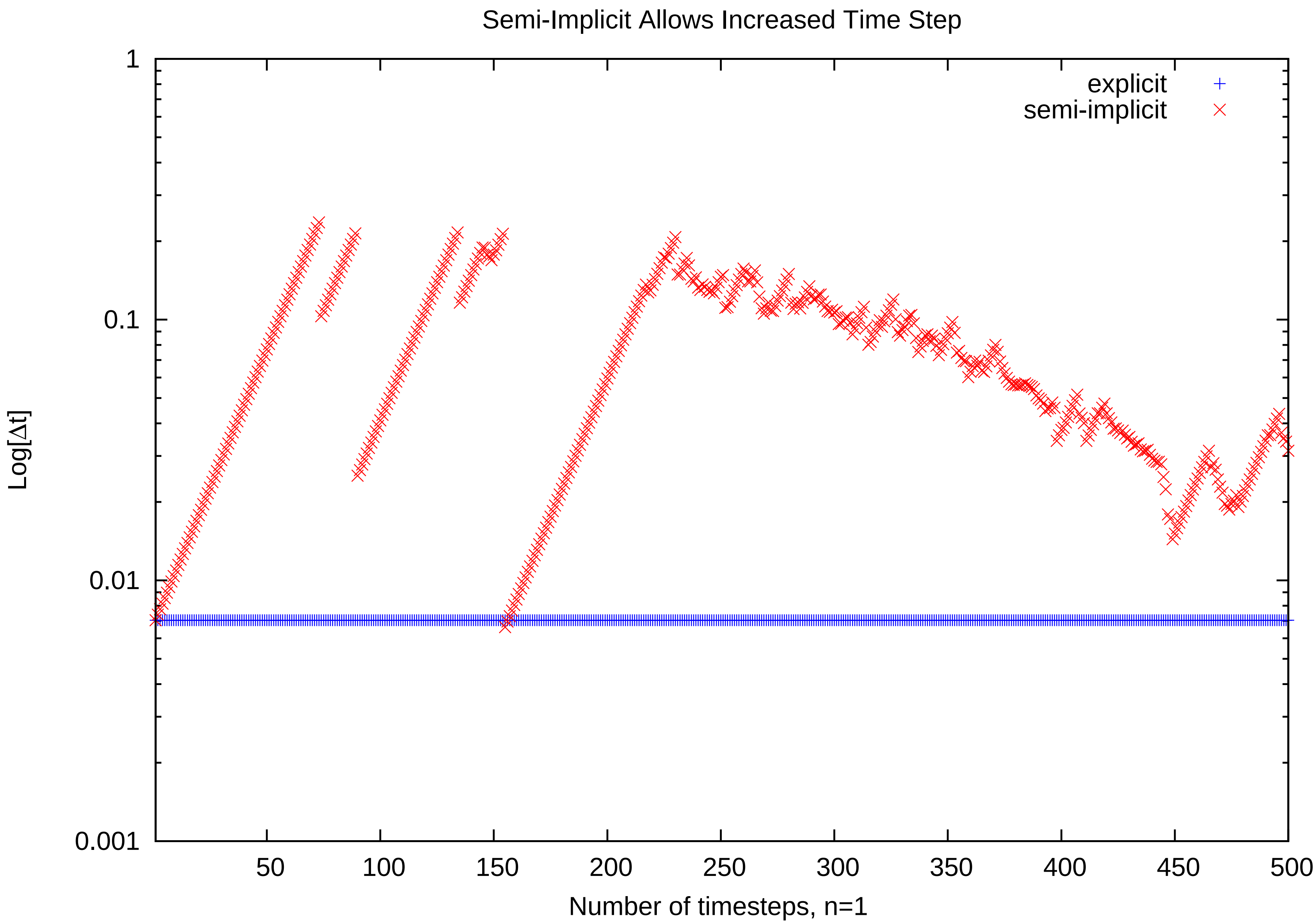


For $k = \frac{2\pi n}{L}$, where $L = 10, \gamma = \frac{5}{3}, P_0 = 1, \rho_0 = 1$

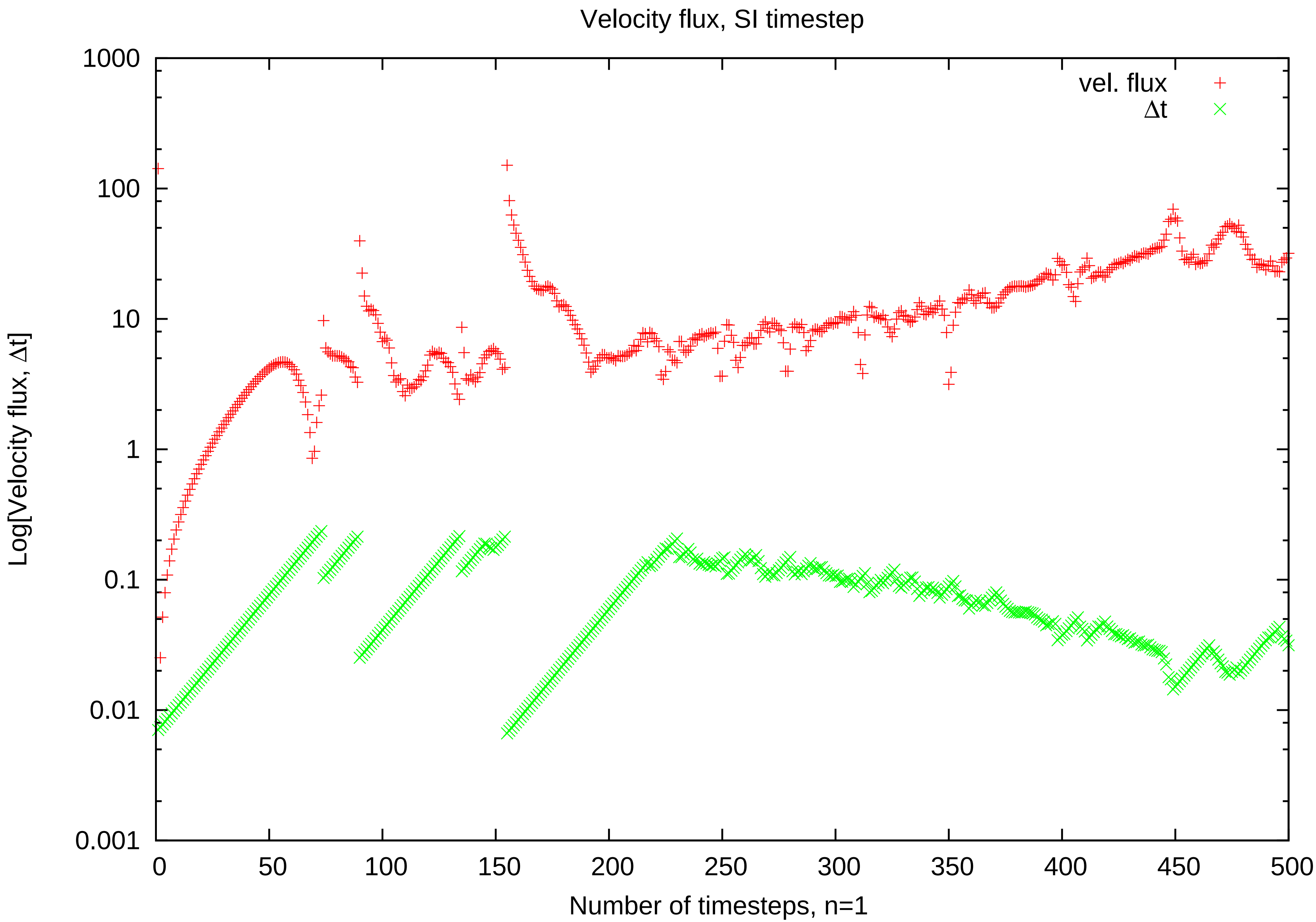
MH4D Resolves Linear Sound Wave



Advantage of Semi-Implicit



Adjusting for Fluid Flow



Linear Shear Alfvén Wave

Assume: $\mathbf{v}_0 = \mathbf{0}, \mathbf{B}_0 = B_0\hat{\mathbf{y}}$, and $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B}$

Linear system:

$$\frac{\partial \mathbf{A}_1}{\partial t} = \mathbf{v}_1 \times \mathbf{B}_0$$

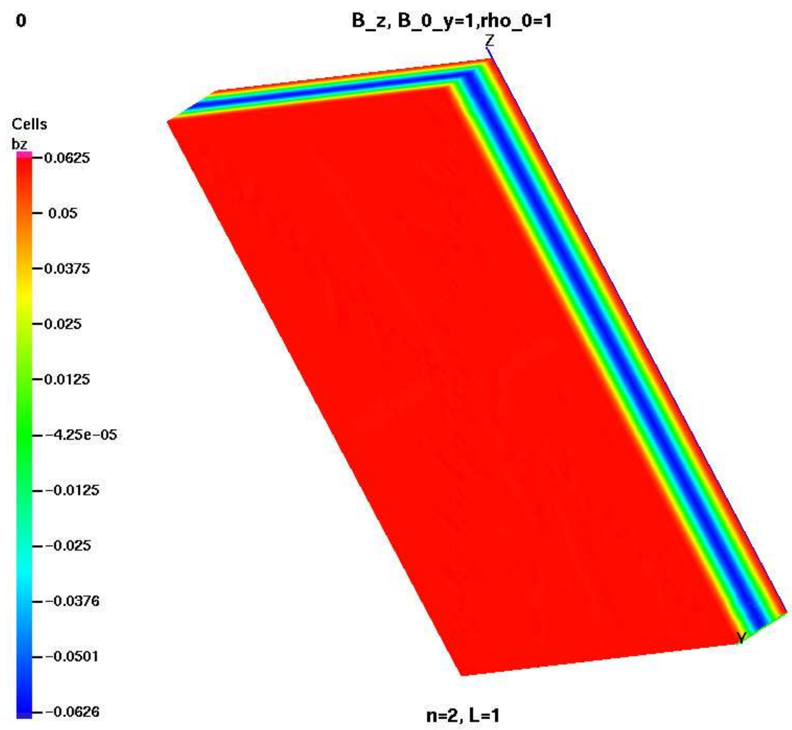
$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{J}_1 \times \mathbf{B}_0$$

Eigenvalue: $\omega = k_y \frac{B_0}{\sqrt{\rho_0}} = k_y V_A$

Eigenvectors:

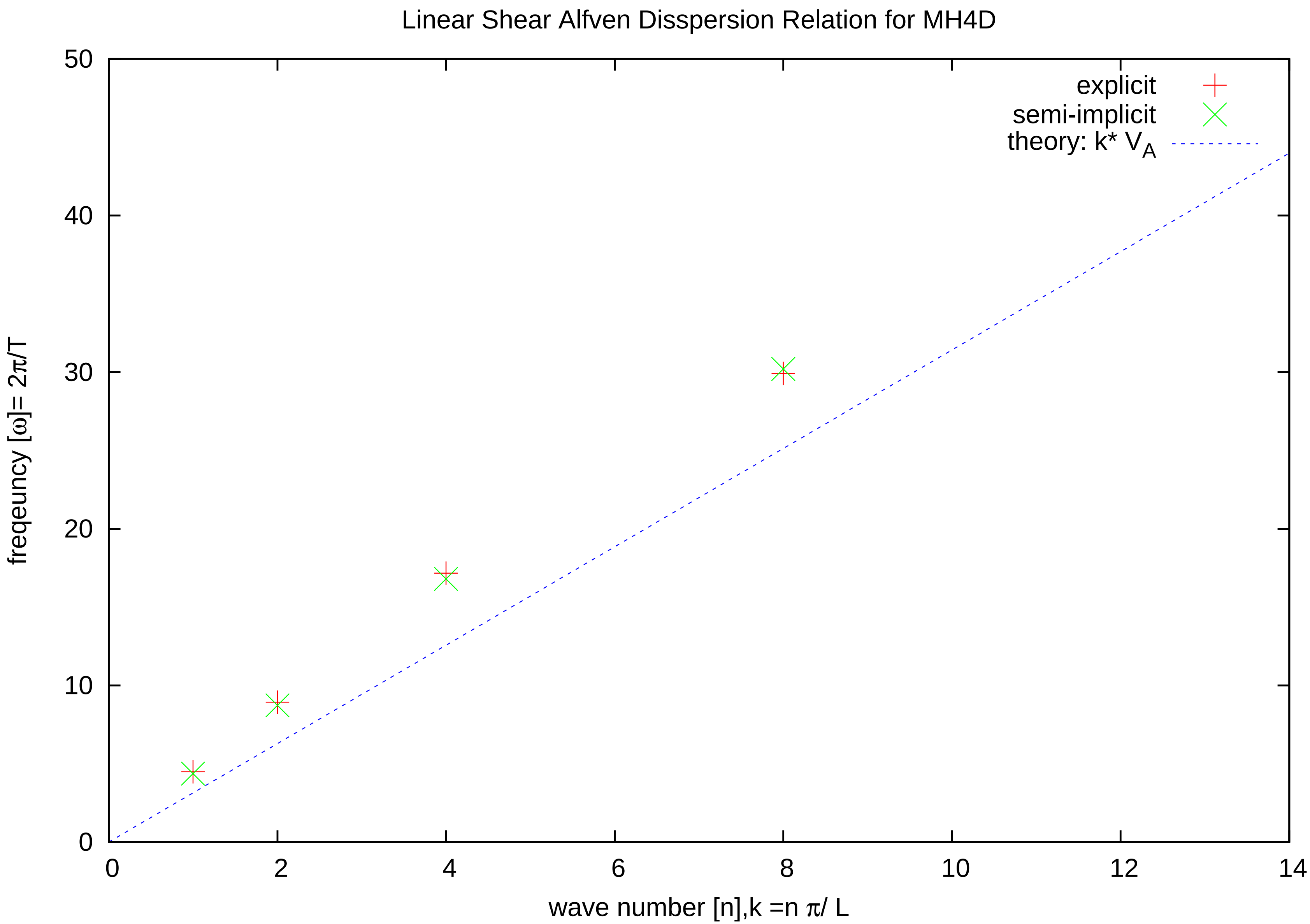
$$\mathbf{A}_1 = -\epsilon \rho_0 \sin(k_y y) \cos(\omega t) \hat{\mathbf{x}}$$

$$\mathbf{v}_1 = -\epsilon k_y \sqrt{\rho_0} \sin(k_y y) \sin(\omega t) \hat{\mathbf{z}}$$



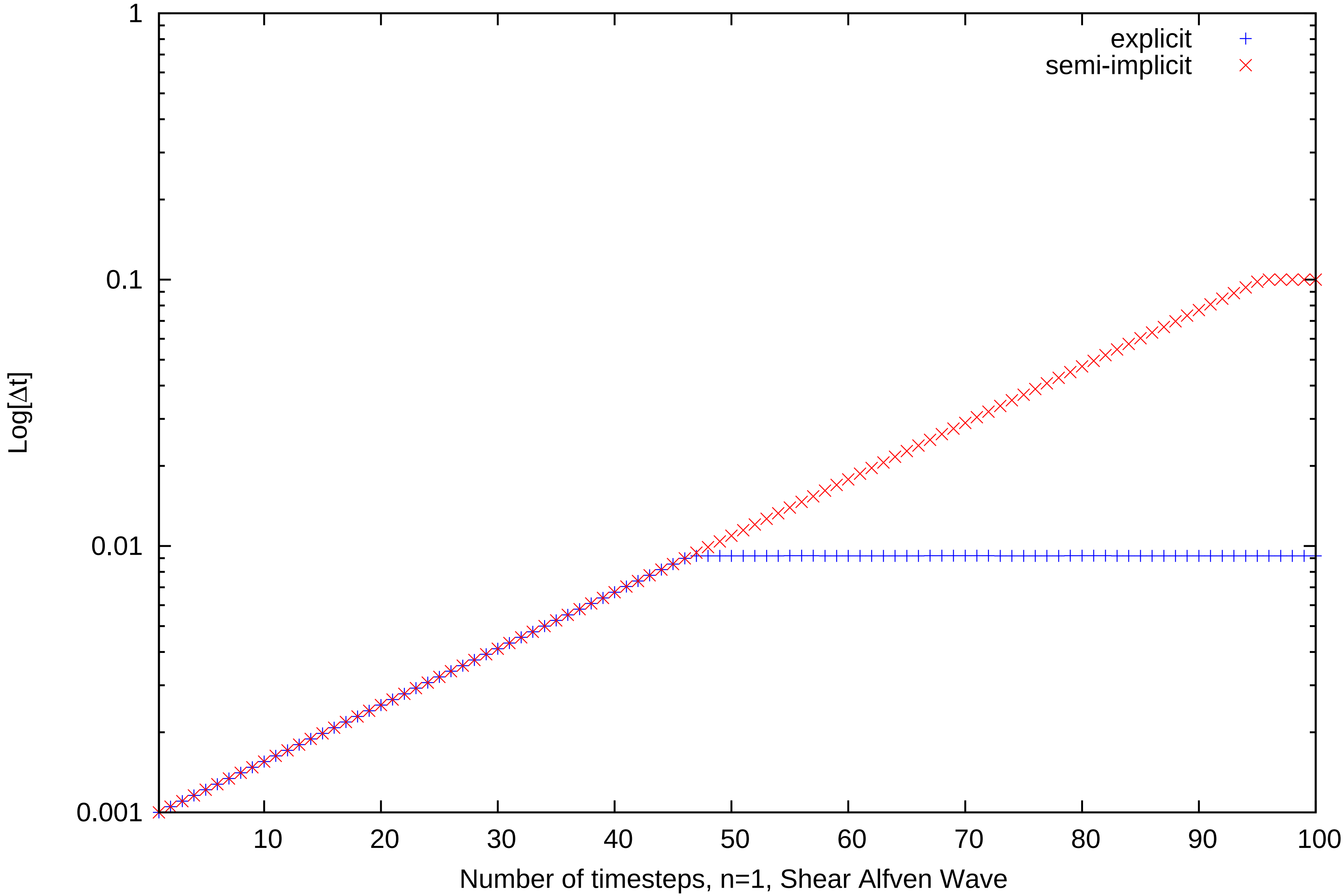
Let $\mathbf{A}_0 = \frac{B_0}{2}(z\hat{\mathbf{x}} - x\hat{\mathbf{z}})$, and $k_y = \frac{\pi n}{L}$, where $L = 1, B_0 = 1, \rho_0 = 1$.

MH4D Resolves the Shear Alfvén Wave



Advantage of Semi-Implicit

Semi-Implicit Allows Increased Time Step



Conclusion

The PSI-Center is refining the tetrahedral mesh MHD simulation code, MH4D, to model Innovative Confinement Concepts experiments.

Semi-Implicit Time Stepping :

- succesful for linear sound wave
- promising results for linear shear Alfvén wave

Current challenges:

- test semi-implicit for non-linear MHD problem
- complicated boundary conditions

Future challenges:

- regions of near zero density
- neutral gas injection and ionization

References

[1] R. Lionello, Z. Mikić, and D. D. Schnack. Magnetohydrodynamics of Solar Coronal Plasmas in Cylindrical Geometry. *Journal of Computational Physics*, 140:1–30, January 1998.

[2] R. Lionello, Z. Mikić, and J. A. Linker. Stability of Algorithms for Waves with Large Flows. *Journal of Computational Physics*, 152:346–358, June 1999.

[3] D. S. Harned and D. D. Schnack. Semi-implicit method for long time scale magnetohydrodynamics computations in three dimensions. *J. Comput. Phys.*, 65:57, 1986.

[4] D. D. Schnack, D. C. Barnes, Z. Mikić, and E. J. Carmana. Semi-implicit magnetohydrodynamics calculations. *J. Comput. Phys.*, 70:330, 1987.