Open Boundary Condition Progress

E.T. Meier, A.H. Glasser, W. Lowrie,
V.S. Lukin, U. Shumlak, G. Marklin

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Motivation:

- Artificial boundaries are needed to truncate the computational domain without disruptive boundary effects.
  - E.g. thruster models, magnetic reconnection problems.
- Hyperbolic-based open BC are successfully implemented, but fail for practical MHD computation with SEL/HiFi where dissipation is necessary.

Outline:

- Hyperbolic-based open BC concepts
- Ideal MHD results
- Theoretical basis for open BC in dissipative systems.

Hyperbolic-based open BC concept

- Boundary conditions influence the interior solution through the boundary flux

$$\int_V \left(\partial_t \mathbf{Q} + \nabla \bullet \mathbf{F}\right) = 0$$

$$\int_V \partial_t \mathbf{Q} + \int_S \mathbf{F} = 0$$

- Characteristic decomposition of hyperbolic systems is used to properly treat individual waves.

$$\frac{\partial \vec{\mathbf{u}}}{\partial (time)} + \nabla \bullet \overleftrightarrow{\mathbf{F}} = 0 \quad \rightarrow \quad \frac{\partial \vec{\mathbf{u}}}{\partial (time)} + \overleftrightarrow{\mathbf{A}}_n \frac{\partial}{\partial n} \vec{\mathbf{u}} + \overleftrightarrow{\mathbf{A}}_t \frac{\partial}{\partial t} \vec{\mathbf{u}} = 0$$

Approximate Riemann BC approach specifies ambient conditions.

- Treat boundary like a discontinuity and solve the Riemann problem using Roe's method. $\frac{\partial \vec{\mathbf{u}}}{\partial(time)} + \frac{\partial}{\partial n}\left(\vec{\mathbf{F}}_n^{\,+} + \vec{\mathbf{F}}_n^{\,-}\right) + \overleftrightarrow{\mathbf{A}}_t \frac{\partial}{\partial t}\vec{\mathbf{u}} = 0$

$$\vec{\mathbf{F}}_n \cong \overleftrightarrow{\mathbf{A}}_n^{*}\mathbf{u}$$

- $\overleftrightarrow{\mathbf{A}}_n^{*}$ is determined by averaging the conditions on either side of the discontinuity.

Thompson* presents an approach derived to achieve non-reflection.

- Consider the 1D wave equation,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 .$$

- Non-reflection requires $c\frac{\partial u}{\partial x} = 0$.

- A hyperbolic system can be written as *n* independent wave-like equations,

$$\vec{\mathbf{l}}_i \frac{\partial}{\partial(time)}\vec{\mathbf{u}} + \lambda_i \vec{\mathbf{l}}_i \frac{\partial}{\partial n}\vec{\mathbf{u}} + \vec{\mathbf{l}}_i \left(\overleftrightarrow{\mathbf{A}}_t \frac{\partial}{\partial t}\vec{\mathbf{u}} \right) = 0;\ i = 1,\ldots,n .$$

- By analogy, non-reflection requires

$$\lambda_i \vec{\mathbf{l}}_i \frac{\partial}{\partial n}\vec{\mathbf{u}} = 0 \text{ for } \lambda_i < 0 .$$

* **K.W. Thompson**, *Time dependent boundary conditions for hyperbolic systems*, J. Comp. Phys., 68 (1987), 1

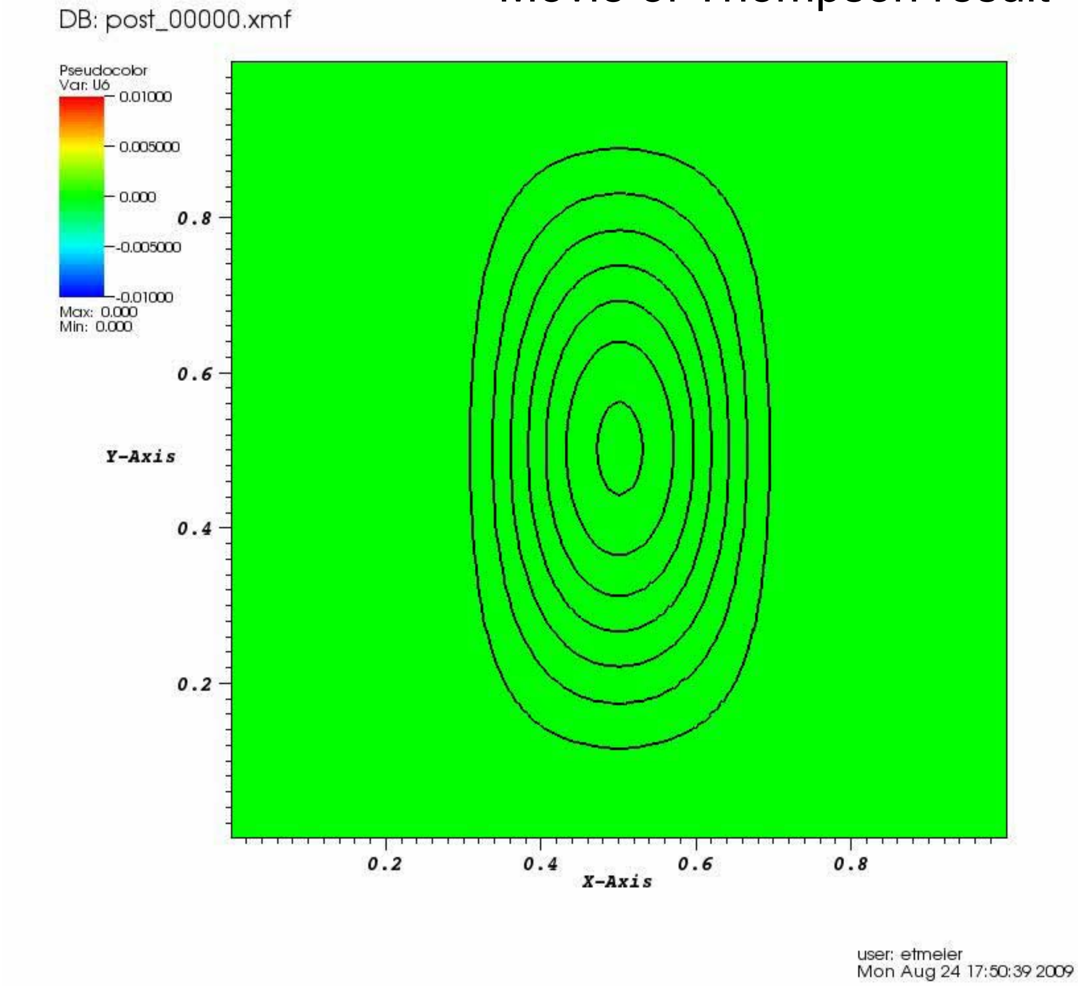
As an ideal MHD open BC, Thompson approach gives smooth solution.

Initial condition is a 2x overpressure.

Contours are of internal energy.

Pseudocolor is By.

Movie of Thompson result



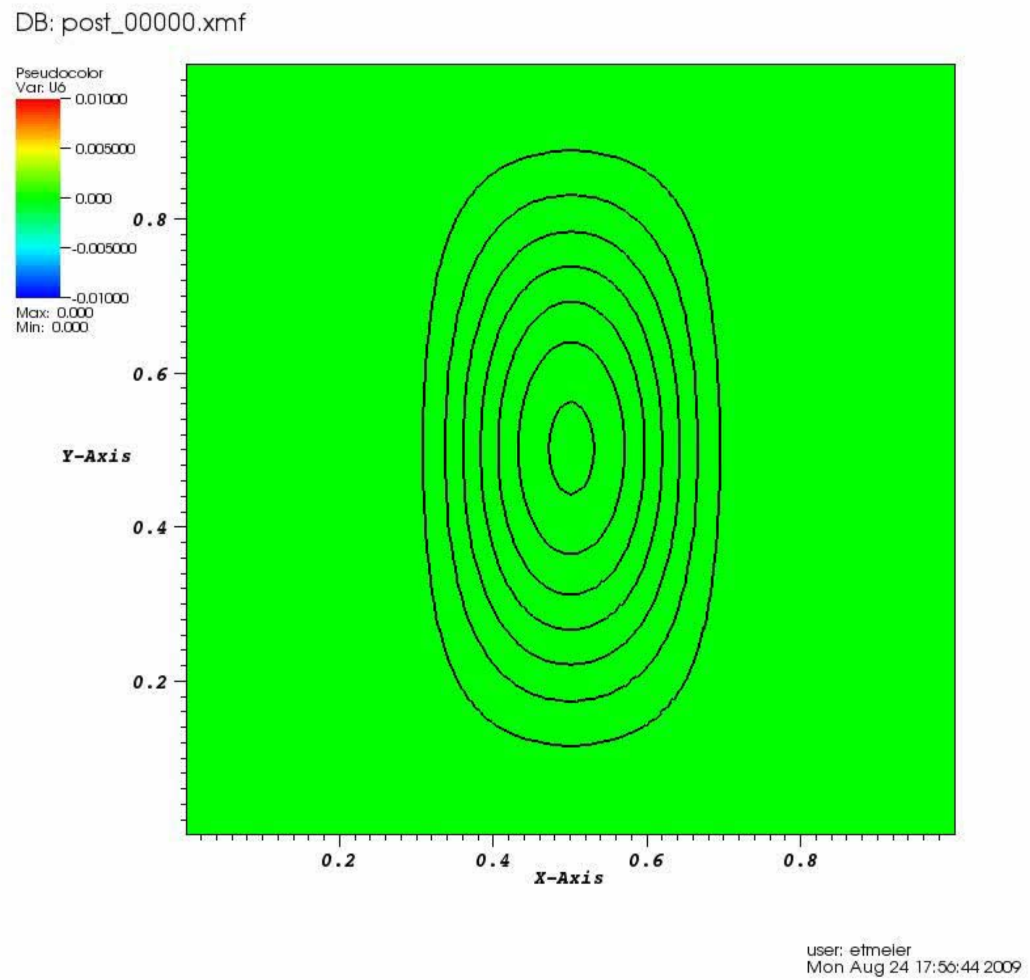
As an ideal MHD open BC, Approximate Riemann approach gives rough solution.

Initial condition is a 2x overpressure.

Contours are of internal energy.

Pseudocolor is By.

Movie of A.R. result

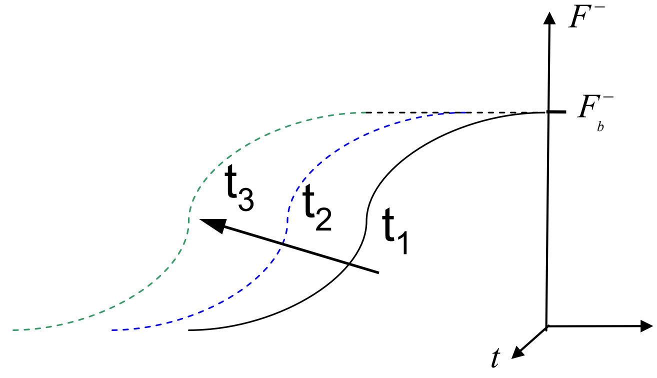


Do tangential dynamics cause boundary layers in Approximate Riemann BC approach?

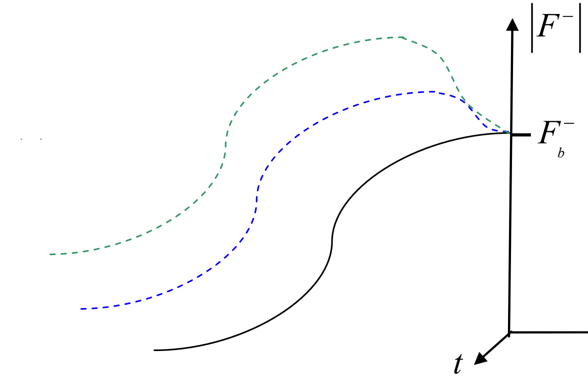
AR BC

$$F_b^- = g$$

a) There is no tangential variation. The incoming wave matches the interior.



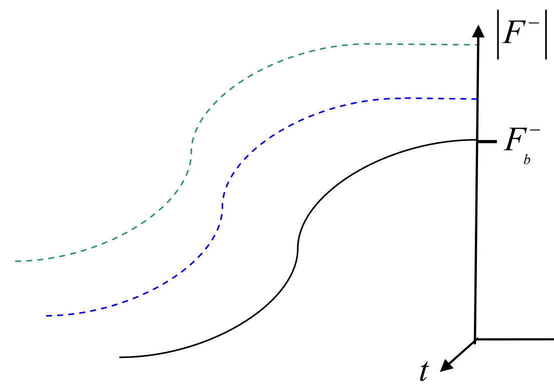
b) Tangential variation changes the wave flux near the boundary. The incoming wave doesn’t match the interior.



Thompson BC

$$\frac{\partial}{\partial n} F_b^- = 0$$

c) Normal derivative of flux is zero and boundary flux matches interior even with tangential variation.



Dissipative effects further complicate matters.

	AR	Thompson
Euler	✓	✓
Navier-Stokes	✓	✓
Ideal MHD	✗	✓
Dissipative MHD	✗	✗

Well-posed BC can be derived via energy analysis.

- Examples of well-posed BC derivations using energy analysis exist for Navier-Stokes.*
- Can an energy analysis approach be applied to dissipative MHD?
  - Preliminary thoughts are on next slides.

* **J.S. Hesthaven and D. Gottlieb**, *A stable penalty method for the compressible Navier-Stokes equations: I. Open boundary conditions*, SIAM J. Sci. Comput., 17 (1996) 579
* **J. Nordstrom and M. Svard**, *Well-posed boundary conditions for the Navier-Stokes equations*, SIAM J. Numer. Anal., 43 (2005) 1231

Energy analysis (1)

- Energy analysis provides insight.
  - We'll see the connection to hyperbolic AR and Thompson.
  - Could provide a basis for dissipative MHD open BC.
- Begin with restricted conservation form (no sources):

$$\frac{\partial u^{\mu}}{\partial t}+\frac{\partial F_i^{\mu}}{\partial x_i}=0,\quad F_i^{\mu}=F_i^{\mu}\left(u^{\nu},u_{x_j}{}^{\nu}\right)$$

$$\frac{\partial u^{\mu}}{\partial t}+A_{i\nu}^{\mu}\frac{\partial u^{\nu}}{\partial x_i}=B_{ij\nu}^{\mu}\frac{\partial^2 u^{\nu}}{\partial x_i\partial x_j}$$

$$A_{i\nu}^{\mu}\equiv\frac{\partial F_i^{\mu}}{\partial u^{\nu}},\quad B_{ij\nu}^{\mu}\equiv-\frac{\partial F_i^{\mu}}{\partial u_{x_j}^{\nu}}$$

Energy analysis (2)

- Symmetric matrices are required for later steps.
  - Three **A** matrices and nine **B** matrices must be simultaneously symmetrized.
  - This might be impossible for MHD (even if we ignore the **B**'s).
- Create new variable, **s**=**S**$^{-1}$**u**. (This requires linearization.)

$$\overline{A}^{\mu}_{i\nu} \cong A^{\mu}_{i\nu}, \quad \overline{B}^{\mu}_{ij\nu} \cong B^{\mu}_{ij\nu}$$

$$\hat{A}^{\rho}_{i\sigma} = \left(S^{-1}\right)^{\rho}_{\mu} \overline{A}^{\mu}_{i\nu} S^{\nu}_{\sigma}, \quad \hat{B}^{\rho}_{ij\sigma} = \left(S^{-1}\right)^{\rho}_{\mu} \overline{B}^{\mu}_{ij\nu} S^{\nu}_{\sigma}$$

$$\frac{\partial u^{\mu}}{\partial t} + S^{\mu}_{\rho} \hat{A}^{\rho}_{i\sigma} \left(S^{-1}\right)^{\sigma}_{\nu} \frac{\partial u^{\nu}}{\partial x_i} = S^{\mu}_{\rho} \hat{B}^{\rho}_{ij\sigma} \left(S^{-1}\right)^{\sigma}_{\nu} \frac{\partial^2 u^{\nu}}{\partial x_i \partial x_j}$$

$$\frac{\partial s^{\mu}}{\partial t} + \hat{A}^{\mu}_{i\nu} \frac{\partial s^{\nu}}{\partial x_i} = \hat{B}^{\mu}_{ij\nu} \frac{\partial^2 s^{\nu}}{\partial x_i \partial x_j}$$

Energy analysis (3)

- Energy integral – take product with **s** and integrate over volume.
- Results in interior terms and surface terms.
- We want to apply BCs to the surface terms.

$$E \equiv \int_{\Omega} d\mathbf{x}\ \mathrm{s}_{\mu} \left(\frac{\partial s^{\mu}}{\partial t} + \hat{A}^{\mu}_{i\nu} \frac{\partial s^{\nu}}{\partial x_i} - \hat{B}^{\mu}_{ij\nu} \frac{\partial^2 s^{\nu}}{\partial x_i \partial x_j} \right)$$

$$= \frac{1}{2} \frac{d}{dt} \int_{\Omega} d\mathbf{x}\ s_{\mu} s^{\mu} + \int_{\Omega} d\mathbf{x} \hat{B}^{\mu}_{ij\nu} \frac{\partial^2 s^{\nu}}{\partial x_i \partial x_j}$$

$$+ \int_{\partial\Omega} d\mathbf{x}\ s_{\mu} n^{i} \left(\frac{1}{2} s_{\mu} \hat{A}^{\mu}_{i\nu} s^{\nu} - s_{\mu} \hat{B}^{\mu}_{ij\nu} \frac{\partial s^{\nu}}{\partial x_j} \right)$$

Energy analysis (4)

- An integral of **vCv** emerges.
  - **v** is a vector of variables and their spatial derivatives
  - **C** is a matrix analogous to **A**, the flux jacobian of a purely hyperbolic system.
- Boundary contribution is controlled by decomposing **C** into **C**=**RDL** and specifying (**L**$^-$)**v**.

$$\int_{\partial\Omega} d\mathbf{x}\; s_\mu n^i \left(\frac{1}{2} s_\mu \hat{A}^{\mu}_{i\nu} s^\nu - s_\mu \hat{B}^{\mu}_{ij\nu} \frac{\partial s^\nu}{\partial x_j} \right)$$

$$= \frac{1}{2} \int_{\partial\Omega} d\mathbf{x}\; v_\mu C^\mu_\nu v^\nu$$

$$\mathbf{v} = \begin{pmatrix} s^\mu \\ \dfrac{\partial s^\nu}{\partial x_j} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} n^i \hat{A}^{\mu}_{i\nu} & n^i \hat{B}^{\mu}_{ij\nu} \\ n^i \hat{B}^{\nu}_{ij\mu} & 0\delta^\mu_\nu \end{pmatrix}$$

Energy analysis conclusions / questions

- Analysis says that BC should be set on $\left(\mathbf{L}^{-}\right)\mathbf{v}$.
  - This is consistent with the AR approach.

- Is a BC on $\left(\mathbf{L}^{-}\right)\dfrac{\partial}{\partial n}\mathbf{v}$ appropriate?
  - This would correspond to the Thompson approach.

- What about simultaneous symmetrization?
  - It might not matter.
  - In the Thompson and AR BCs, which work for many problems, the anti-symmetric contributions don't seem problematic.

Future work

- Find proper **C** matrix for dissipative MHD.
- Identify proper BC: $\mathbf{F}^- = \mathbf{g}$ vs. $\dfrac{\partial}{\partial n}\mathbf{F}^- = 0$.
- Implement and test.